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Letter to the Editor

A Note on Rational Approximation on $[0, \infty)$

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In this note we prove the following theorem.

THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(a_0 > 0,\ a_k \geq 0$ *for all* $k \geq 1)$ *be an entire function of order* $\rho(0 < \rho < \infty)$, *type* τ *and lower type* $\omega(0 < \omega \leq \tau < \infty)$. *Then for any sequence* $P_n(x)$ *of polynomials of degree at most* n, *positive on* $[0, \infty)$, *we have*

$$\liminf_{n \to \infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geq \left(4 \left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right)^{-2}. \tag{1}$$

Remark. This extends a result of Meinardus and Varga [2, Theorem 3] and of Reddy [3, Theorem D].

Proof. Let $\epsilon > 0$. For all $r \geq$ some $r_0(\epsilon)$, we have

$$\omega(1 - \epsilon)\, r^\rho \leq \log M(r) \leq \tau(1 + \epsilon)\, r^\rho, \tag{2}$$

where $M(r) = \operatorname{Max}_{|z| \leq r} |f(z)|$.
From (2) it is easy to get for any $\delta > 1$ and all large r,

$$M(r\delta) \geq \{M(r)\}^{(\delta^\rho \omega(1-\epsilon)/\tau(1+\epsilon))}. \tag{3}$$

Now let us assume (1) is false. Then there exists an infinite sequence of integers, $1 \leq n_1 < n_2 < n_3 < \cdots$, such that, for $q = 1, 2, 3, \ldots$,

$$\left\| \frac{1}{f(x)} - \frac{1}{P_{n_q}(x)} \right\|_{L_\infty[0,\infty)} < \left(4 \left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right)^{-2n_q}. \tag{4}$$

Since $\lim_{x\to\infty} f(x) = \infty$, for all large n there is an $r_n > 0$ such that

$$f(r_n) = \left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - \frac{5}{4}\right)^n. \tag{5}$$

Then from (4) and (5) we obtain, for all large q,

$$P_{n_q}(r_{n_q}) < \left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right)^{n_q}. \tag{6}$$

Otherwise, (4) would be contradicted. Set $\delta = (2\tau/\omega)^{1/\rho}$; then from (3) and (5) we get, for all large q,

$$f(r_{n_q}\delta) \geqslant \{f(r_{n_q})\}^{2(1-\epsilon)/(1+\epsilon)} = \left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - \frac{5}{4}\right)^{2n_q(1-\epsilon)/(1+\epsilon)} = g(q). \tag{7}$$

But at $x = r_n\delta$, by using a result of Remez [1, pp. 534–35] along with (6), we get for all large q,

$$P_{n_q}(r_{n_q}\delta) < \left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right)^{n_q}\left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - 2\right)^{n_q} = h(q). \tag{8}$$

From (7) and (8) we get, for all large q, ϵ being arbitrary,

$$\left(4\left(\frac{2\tau}{\omega}\right)^{1/\rho} - 1\right)^{-2n_q} < \frac{1}{h(q)} - \frac{1}{g(q)} < \frac{1}{P_{n_q}(r_{n_q}\delta)} - \frac{1}{f(r_{n_q}\delta)},$$

which contradicts (4), hence the result.

References

1. P. Erdös and P. Turán, On interpolation III, Interpolatory theory of polynomials, *Ann. of Math.* **41** (1940), 510–553.
2. G. Meinardus and R. S. Varga, Chebyshev rational approximation to certain entire functions in $[0, \infty)$, *J. Approximation Theory* **3** (1970), 300–309.
3. A. R. Reddy, A contribution to rational Chebyshev approximation to certain entire functions in $[0, \infty)$, *J. Approximation Theory*, **11** (1974), 85–96.